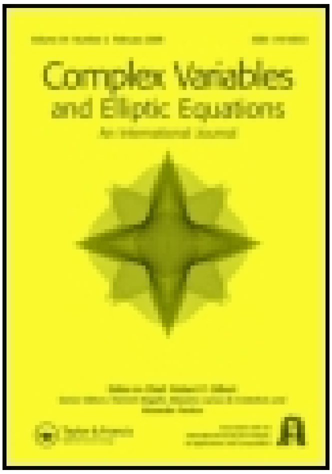

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Linear Fractional Transformation Methods in $ {\shadC}^n $

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Linear Fractional Transformation Methods in $\mathbb{C}^n$

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We consider the generalization of linear fractional transformations of the plane to $\mathbb{C}^n$. Analogs of the one-variable theory are developed including fixed point sets and points of symmetry. The domains in $\mathbb{C}^n$ that are images of the ball under these transformations are found. Finally, we see some examples where classical fixed point results follow from this theory in a natural way.

Keywords: Biholomorphic; Convex mapping; Holomorphic automorphism

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1. INTRODUCTION

The group of linear fractional transformations of the complex plane,

$$T(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

is a useful tool in the theory of functions of one complex variable. In this article, we will see that the natural generalizations of these functions have similarly useful properties in the setting of several complex variables.

We will work in complex Euclidean space $\mathbb{C}^n$, consisting of vectors of the form $z = (z_1, \ldots, z_n)$ and equipped with the standard Hermitian inner product $\langle z, w \rangle = \sum_{k=1}^{n} z_k \overline{w}_k$ and associated norm $\|z\| = \langle z, z \rangle^{1/2}$ for $z, w \in \mathbb{C}^n$. When used in computation with matrices, $z \in \mathbb{C}^n$ will always be treated as a column vector. Let $B = B_n = \{z \in \mathbb{C}^n\colon \|z\| < 1\}$ denote the open unit ball, and set $e_1, \ldots, e_n$ to be the standard basis vectors. If $z \in \mathbb{C}^k$ and $w \in \mathbb{C}^{n-k}$ with $k = 1, \ldots, n-1$, write (z, w) for the vector in $\mathbb{C}^n$ whose first k components are z and remaining $n-k$ components are w. Note that the word "dimension" will always mean as a complex space.

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Let $L(\mathbb{C}^n)$ denote the algebra of linear operators (realized as $n \times n$ matrices) from $\mathbb{C}^n$ into $\mathbb{C}^n$ with identity I, and $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ be the general linear and special linear groups, respectively. That is, $A \in \mathrm{GL}(n, \mathbb{C})$ provided that $\det A \neq 0$, and $A \in \mathrm{SL}(n, \mathbb{C})$ if $\det A = 1$. For any complex matrix A, the adjoint (conjugate-transpose) is given by A^*. Set $\mathcal{U}(n) \subseteq L(\mathbb{C}^n)$ to be the group of unitary operators. In other words $U \in \mathcal{U}(n)$ if and only if $U^* = U^{-1}$.

Complex projective n-space $\mathbb{CP}^n$ is the complex manifold realized as the set of all complex lines in $\mathbb{C}^{n+1}$. Formally, $\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$, the set of all equivalence classes under the equivalence relation $\sim$, defined for $z, w \in \mathbb{C}^{n+1} \backslash \{0\}$ by $z \sim w$ if and only if $z = \lambda w$ for some $\lambda \in \mathbb{C} \backslash \{0\}$. Denote the equivalence class under $\sim$ of a given $z \in \mathbb{C}^{n+1} \backslash \{0\}$ by $[z] = [z_1 : \cdots : z_{n+1}]$. We consider $\mathbb{C}^n \subseteq \mathbb{CP}^n$ by the embedding $z \mapsto [z : 1]$ (with the obvious simplification in notation).

The group $\mathrm{Aut}\, B$ of biholomorphic automorphisms of B are well-known examples of the functions we will study. Recall that $T \in \mathrm{Aut}\, B$ if and only if $T = U \circ \varphi_{ru}$, where $U \in \mathcal{U}(n)$ and $\varphi_{ru} : B \to B$ is the involution given by

$$\varphi_{ru}(z) = \frac{ru - P_u z - s_r Q_u z}{1 - r\langle z, u \rangle}, \quad z \in B, \tag{1.1}$$

with $r \in [0, 1)$, $u \in \partial B$, $s_r = \sqrt{1 - r^2}$, and projections $P_u = \langle \cdot, u \rangle u$ and $Q_u = I - P_u$. Another example of these functions is the Cayley transform

$$T(z) = \frac{e_1 + z}{1 - z_1}, \tag{1.2}$$

which maps B onto the Siegel generalized right half-space

$$H = \left\{ z \in \mathbb{C}^n \colon \mathrm{Re}\, z_1 > \sum_{k=2}^{n} |z_k|^2 \right\}. \tag{1.3}$$

These functions are thoroughly discussed in [6].

After defining the group of generalized linear fractional transformations in $\mathbb{C}^n$, we will see that these functions extend naturally to homeomorphisms of $\mathbb{CP}^n$. The study of fixed points leads to an understanding of how many points (and of what type) will uniquely determine a transformation. We find the orbit of the ball B under the action of the transformation group and see how the concept of symmetric points can be used to map one of these domains to another. We will conclude by showing how an approach using linear fractional transformations in $\mathbb{C}^n$ provides results concerning fixed boundary points of automorphisms of B and H and the convergence of iterates of these maps to the boundary. This approach mimics the familiar proofs of the analogous results in the one-variable case.

2. LINEAR FRACTIONAL TRANSFORMATIONS IN $\mathbb{C}^n$

We begin by considering linear fractional functions of the form

$$T(z) = \frac{Az + a}{\alpha + \langle z, b \rangle}, \tag{2.1}$$

where $A \in L(\mathbb{C}^n)$, $a, b \in \mathbb{C}^n$, and $\alpha \in \mathbb{C}$, defined for all $z \in \mathbb{C}^n$ such that $\langle z, b \rangle \neq -\alpha$. Note that T is simply the ratio of an affine mapping with an affine functional. If we take another such function

$$S(z) = \frac{Bz + c}{\beta + \langle z, d \rangle},$$

then we see that the composition

$$T \circ S(z) = \frac{(AB + ad^*)z + (Ac + \beta a)}{(\alpha\beta + \langle c, b \rangle) + \langle z, B^* b + \overline{\alpha} d \rangle}$$

is another function of the same type, taking into consideration obvious domain constraints.

Associate to T and S the operators in $L(\mathbb{C}^{n+1})$ given in block matrix form by

$$\tilde{T} = \begin{bmatrix} A & a \\ b^* & \alpha \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} B & c \\ d^* & \beta \end{bmatrix}. \tag{2.2}$$

A simple calculation reveals that the product $\tilde{T}\tilde{S}$ corresponds to $T \circ S$ in a similar way. Since any operator in $L(\mathbb{C}^{n+1})$ corresponds to a linear fractional function in the sense of (2.1) and (2.2), a linear fractional function will have an inverse if and only if its associated operator in $L(\mathbb{C}^{n+1})$ is invertible, and its inverse is another linear fractional function. We now have the following definition.

Definition 2.1 A function of the form (2.1) is called a *linear fractional transformation* of $\mathbb{C}^n$ provided that the associated operator given in (2.2) lies in $\mathrm{GL}(n+1, \mathbb{C})$. Denote the set of all linear fractional transformations by $\mathfrak{L}(\mathbb{C}^n)$.

With the obvious domain constraints aside, $\mathfrak{L}(\mathbb{C}^n)$ is a group, under composition, of holomorphic functions. It is known [6] that such functions will map affine sets onto affine sets. Note that it is not necessary that the operator A in (2.1) be invertible in order for T to lie in $\mathfrak{L}(\mathbb{C}^n)$.

If $T \in \mathfrak{L}(\mathbb{C}^n)$, there is not a unique $\tilde{T}$ associated to T in the sense of (2.2). For if $\lambda \in \mathbb{C} \backslash \{0\}$, then $\lambda \tilde{T}$ will associate to T if $\tilde{T}$ does. This exposes the group structure of $\mathfrak{L}(\mathbb{C}^n)$.

THEOREM 2.2 *The groups* $\mathfrak{L}(\mathbb{C}^n)$ *and* $\mathrm{SL}(n+1, \mathbb{C})/Z$ *are isomorphic, where* $Z = \{e^{2k\pi i/(n+1)} I\colon k = 0, \ldots, n\}$ *is the center of* $\mathrm{SL}(n+1, \mathbb{C})$.

As a result, we will always assume that $\tilde{T} \in \mathrm{SL}(n+1, \mathbb{C})$.

Linear fractional transformations of the complex plane extend to homeomorphisms of the extended complex plane $\mathbb{C}_\infty$. Projective space $\mathbb{CP}^n$ is the natural compactification of $\mathbb{C}^n$ for our purposes, as indicated by the following theorem. (Of course, $\mathbb{CP}^1$ is homeomorphic to the sphere S^2, and thus to $\mathbb{C}_\infty$.)

THEOREM 2.3 *If* $\tilde{T} \in \mathrm{SL}(n+1, \mathbb{C})$, *then the function* $T : \mathbb{CP}^n \to \mathbb{CP}^n$ *given by*

$$T([z]) = [\tilde{T} z], \quad z \in \mathbb{C}^{n+1} \backslash \{0\}$$

is a homeomorphism, and $T|_{\mathbb{C}^n}$ *is the member of* $\mathfrak{L}(\mathbb{C}^n)$ *that corresponds to* $\tilde{T}$.

Proof It is easy to see that T is well-defined by observing that the sets $\tilde{T}([z])$ and $[\tilde{T}z]$ are equal for all $z \in \mathbb{C}^{n+1}\backslash\{0\}$. Clearly, T is a homeomorphism. If $\tilde{T}$ is written as in (2.2), then

$$T([z:1]) = [\tilde{T}(z,1)] = \left[\frac{Az+a}{\alpha + \langle z, b\rangle} : 1\right]$$

for all $z \in \mathbb{C}^n$ such that $\langle z, b\rangle \neq -\alpha$. ■

If $T \in \mathfrak{L}(\mathbb{C}^n)$ has the form (2.1) and $z \in \mathbb{C}^n$ is such that $\langle z, b\rangle = -\alpha$, then $T(z)$ can be realized as a point of $\mathbb{CP}^n\backslash\mathbb{C}^n$. For this reason, we define the *set*

$$\infty = \mathbb{CP}^n\backslash\mathbb{C}^n.$$

Either T maps an $(n-1)$-dimensional affine subset of $\mathbb{C}^n$ into ∞ or T is an affine transformation of $\mathbb{C}^n$. Furthermore, if $n \geq 2$, then one point of ∞ must map into ∞ because ∞ is compact (it is homeomorphic to $\mathbb{CP}^{n-1}$) and $T(\infty) \cap \mathbb{C}^n$ is an $(n-1)$-dimensional affine set.

3. FIXED POINT SETS

In the plane, a linear fractional transformation is uniquely determined by its action on three points of $\mathbb{C}$. We will see that, with some restrictions, a similar result holds in $\mathbb{C}^n$. We begin with a lemma.

LEMMA 3.1 *If $T \in \mathfrak{L}(\mathbb{C}^n)$ fixes the points $[e_k : 1]$ for $k = 1, \ldots, n$, $[0 : \cdots : 0 : 1]$, and $[1 : \cdots : 1 : 0]$ in $\mathbb{CP}^n$, then T is the identity.*

Proof If $\tilde{T} \in \mathrm{SL}(n+1, \mathbb{C})$ corresponds to T, then $(e_k, 1)$, $k = 1, \ldots, n$, $(0, \ldots, 0, 1)$, and $(1, \ldots, 1, 0)$ are eigenvectors of $\tilde{T}$, hence $\tilde{T}$ is a multiple of I. ■

THEOREM 3.2 *Let $X = \{z_1, \ldots, z_{n+2}\}$ and $Y = \{w_1, \ldots, w_{n+2}\}$ be sets of nonzero vectors in $\mathbb{C}^{n+1}$, and suppose that $X\backslash\{z_k\}$ is linearly independent for any $k = 1, \ldots, n+2$. There exists $T \in \mathfrak{L}(\mathbb{C}^n)$ such that*

$$T([z_k]) = [w_k], \quad k = 1, \ldots, n+2, \tag{3.1}$$

in $\mathbb{CP}^n$ if and only if $Y\backslash\{w_k\}$ is linearly independent for each $k = 1, \ldots, n+2$.

Proof Without loss of generality, suppose that $z_k = (e_k, 1)$ for $k = 1, \ldots, n$, $z_{n+1} = (0, \ldots, 0, 1)$, and $z_{n+2} = (1, \ldots, 1, 0)$. If $Y\backslash\{w_k\}$ is linearly independent for all k, then the (block) matrix

$$W = [w_1 \quad \cdots \quad w_{n+1}]$$

is invertible. If some coordinate, say the kth, of $W^{-1}w_{n+2}$ equals 0, then w_{n+2} is a linear combination of $\{w_1, \ldots, w_{n+1}\}\backslash\{w_k\}$, which is impossible. Thus there exist $\lambda_1, \ldots, \lambda_{n+2} \in \mathbb{C}\backslash\{0\}$ such that

$$W^{-1}w_{n+2} = \lambda_{n+2}^{-1}(\Lambda e, \, -n\lambda_{n+1}),$$

where $e = (1, \ldots, 1)$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Define

$$\tilde{T} = W\begin{bmatrix} \Lambda & 0 \\ -\lambda_{n+1}e^T & \lambda_{n+1} \end{bmatrix}. \tag{3.2}$$

Then

$$\tilde{T}\begin{bmatrix} I & 0 \\ e^T & 1 \end{bmatrix} = W\begin{bmatrix} \Lambda & 0 \\ 0^T & \lambda_{n+1} \end{bmatrix}. \tag{3.3}$$

It follows from (3.2) and (3.3) that the transformation $T \in \mathfrak{L}(\mathbb{C}^n)$ corresponding to $\tilde{T}$ satisfies (3.1).

If $S \in \mathfrak{L}(\mathbb{C}^n)$ also satisfies (3.1), then $T \circ S^{-1}$ fixes $[e_k : 1]$ for $k = 1, \ldots, n$, $[0 : \cdots : 0 : 1]$, and $[1 : \cdots : 1 : 0]$. Hence $T = S$ by Lemma 3.1.

The converse follows by simply reversing the above argument. ■

We conclude the section with the following theorem which gives a description of the fixed point set in $\mathbb{C}^n$ of some $T \in \mathfrak{L}(\mathbb{C}^n)$. (We use the notation $\operatorname{Fix} T = \{z \in \mathbb{C}^n\colon T(z) = z\}$.)

THEOREM 3.3 *Let $T \in \mathfrak{L}(\mathbb{C}^n)$. If $\operatorname{Fix} T \neq \emptyset$, then there is a positive integer $m \leq n+1$ such that*

$$\operatorname{Fix} T = \bigcup_{k=1}^{m} E_k, \tag{3.4}$$

where $E_1, \ldots, E_m \subseteq \mathbb{C}^n$ are pairwise disjoint affine sets that satisfy

$$\sum_{k=1}^{m} \dim E_k \leq n - m + 1.$$

Proof Let $\tilde{T} \in \mathrm{SL}(n+1, \mathbb{C})$ correspond to T. If $z \in \operatorname{Fix} T$, then $(z, 1)$ is an eigenvector of $\tilde{T}$. We will thus consider the eigenvalues of $\tilde{T}$.

Set $\sigma(\tilde{T}) \subseteq \mathbb{C}\backslash\{0\}$ to be the set of eigenvalues of $\tilde{T}$. For each $\lambda \in \sigma(\tilde{T})$, let $\tilde{E}_\lambda \subseteq \mathbb{C}^{n+1}$ be the eigenspace of λ. Let Λ be the set of all $\lambda \in \sigma(\tilde{T})$ for which $\tilde{E}_\lambda$ contains vectors with nonzero $(n+1)$th coordinate. (If $\lambda \in \sigma(\tilde{T})\backslash\Lambda$, then $\tilde{E}_\lambda$ will correspond to fixed points in ∞.) It is easy to see that if $z \in \mathbb{C}^n$ is such that $(z, 1) \in \tilde{E}_\lambda$ for some $\lambda \in \Lambda$, then $z \in \operatorname{Fix} T$.

If $\dim \tilde{E}_\lambda = 1$ for some $\lambda \in \Lambda$, then $\tilde{E}_\lambda$ corresponds to a fixed point of T in $\mathbb{C}^n$. Otherwise, suppose that $(z, 1), (w, 1) \in \tilde{E}_\lambda$ for some $\lambda \in \Lambda$ and $z, w \in \mathbb{C}^n$ with $z \neq w$. Then $\mu(z, 1) + \nu(w, 1) \in \tilde{E}_\lambda$ for all $\mu, \nu \in \mathbb{C}$. But if $\mu \neq -\nu$, then

$$[\mu(z, 1) + \nu(w, 1)] = \left[\frac{\mu z + \nu w}{\mu + \nu} : 1\right]$$

shows that the affine combination $(\mu z + \nu w)/(\mu + \nu) \in \operatorname{Fix} T$. It follows that for any $\lambda \in \Lambda$, a basis of the form $\{(z_1, 1), \ldots, (z_k, 1)\}$ (with $z_1, \ldots, z_k \in \mathbb{C}^n$) can be chosen for

$\tilde{E}_\lambda$, and if E_λ is the set of all affine combinations of $z_1, \ldots, z_k$, then $E_\lambda \subseteq \operatorname{Fix} T$. Now distinct members of Λ correspond to disjoint affine sets. Furthermore,

$$\sum_{\lambda \in \Lambda} \dim E_\lambda + \operatorname{card} \Lambda = \sum_{\lambda \in \Lambda} \dim \tilde{E}_\lambda \leq n + 1.$$

Letting $m = \operatorname{card} \Lambda$ gives the result. ∎

4. THE IMAGES OF THE UNIT BALL UNDER TRANSFORMATIONS IN $\mathfrak{L}(\mathbb{C}^n)$

In the complex plane, a linear fractional transformation can be found that will map the open unit disk onto any given interior of a circle, exterior of a circle, or halfplane. Our desire for analogous properties in $\mathbb{C}^n$ motivates us to compute the orbit of B under the action of $\mathfrak{L}(\mathbb{C}^n)$. In other words, what domains in $\mathbb{C}^n$ are the (finite) images of B under transformations in $\mathfrak{L}(\mathbb{C}^n)$? The answer to this question involves the sets H (see (1.3)) and

$$\Omega = \left\{ z \in \mathbb{C}^n \colon |z_1|^2 > 1 + \sum_{k=2}^{n} |z_k|^2 \right\}. \tag{4.1}$$

If $n = 1$, then Ω is the exterior of the unit disk. The following theorem gives properties of Ω for general n.

THEOREM 4.1 *Let $T \in \mathfrak{L}(\mathbb{C}^n)$ be given by*

$$T(z) = \frac{z + (1 - z_1)e_1}{z_1}. \tag{4.2}$$

Then T is a generalization of the inversion in $\mathbb{C}$ and interchanges the sets $\{z \in B\colon z_1 \neq 0\}$ and Ω while mapping $\{z \in \mathbb{C}^n\colon z_1 = 0\}$ into ∞.

Proof All that needs verification is the claim concerning the interchange of $\{z \in B{:}z_1 \neq 0\}$ and Ω. Clearly, the operator $\tilde{T} \in \mathcal{U}(n+1)$ that permutes the first and last coordinates of vectors in $\mathbb{C}^{n+1}$ corresponds to T, and hence T is its own inverse. We thus calculate those $z \in \mathbb{C}^n$ for which $T(z) \in B$.

Observe that

$$\|z + (1 - z_1)e_1\|^2 = \|z\|^2 + 2\operatorname{Re}[(1 - \bar{z}_1)z_1] + |1 - z_1|^2 = \|z\|^2 - |z_1|^2 + 1.$$

Therefore $\|T(z)\| < 1$ if and only if $\|z\|^2 + 1 < 2|z_1|^2$. Subtracting $|z_1|^2$ gives

$$|z_1|^2 > 1 + \sum_{k=2}^{n} |z_k|^2,$$

as desired. ∎

As previously mentioned, our goal is to describe the set

$$\mathcal{A} = \{T(B) \cap \mathbb{C}^n\colon T \in \mathfrak{L}(\mathbb{C}^n)\} \tag{4.3}$$

The following theorem does this in terms of B, H, and Ω.

THEOREM 4.2 *The set $\mathcal{A}$ can be decomposed into disjoint sets as follows*:

$$\mathcal{A} = \mathcal{A}_B \cup \mathcal{A}_H \cup \mathcal{A}_\Omega, \tag{4.4}$$

where $\mathcal{A}_U$ is the collection of full rank affine images of U for $U = B, H, \Omega$.

Proof Clearly, $\mathcal{A}_B$, $\mathcal{A}_H$, and $\mathcal{A}_\Omega$ are disjoint subsets of $\mathcal{A}$, since they contain exclusively bounded domains, unbounded simply connected domains, and domains that are not simply connected, respectively.

Suppose that $U \in \mathcal{A}$ is bounded. Then $U = T(B)$ for some $T \in \mathfrak{L}(\mathbb{C}^n)$ where $T(0) \notin \infty$. Therefore, T may be written as in (2.1) with $\alpha = 1$. Since $T(-b/\|b\|^2) \in \infty$, the boundedness of U forces $\|b\| < 1$. If $b = 0$, then T is a full rank affine mapping. Otherwise, let $\varphi \in \operatorname{Aut} B$ be such that $\varphi(-b) = 0$. Then φ and T map the same $(n-1)$-dimensional affine set into ∞. As a result, the function $F = T \circ \varphi^{-1} \in \mathfrak{L}(\mathbb{C}^n)$ maps ∞ into ∞. Thus F is a full rank affine mapping and $F(B) = U$.

If $U = T(B)$ for some $T \in \mathfrak{L}(\mathbb{C}^n)$ such that $T(u) \in \infty$ for exactly one $u \in \partial B$, then let $L \in \mathcal{U}(n)$ be such that $Le_1 = u$. Clearly, $T \circ L$ takes $\{z \in \mathbb{C}^n: z_1 = 1\}$ into ∞. Let S be given as in (1.2). Then $F = T \circ L \circ S^{-1}$ belongs to $\mathfrak{L}(\mathbb{C}^n)$ and takes ∞ into ∞. Thus F is a full rank affine mapping that takes B onto U.

Lastly, it remains to consider $U = T(B) \cap \mathbb{C}^n$, where $T \in \mathfrak{L}(\mathbb{C}^n)$ sends the $(n-1)$-dimensional affine set $E \subseteq \mathbb{C}^n$ into ∞ and $E \cap B \neq \emptyset$. Let $a \in E \cap B$ and choose $\varphi \in \operatorname{Aut} B$ such that $\varphi(a) = 0$. Now $\varphi(E)$ is an $(n-1)$-dimensional subspace of $\mathbb{C}^n$. Let $L \in \mathcal{U}(n)$ satisfy $L(\varphi(E)) = \{z \in \mathbb{C}^n: z_1 = 0\}$. Then $T \circ \varphi^{-1} \circ L^{-1}$ takes the subspace $\{z \in \mathbb{C}^n: z_1 = 0\}$ into ∞. If S is as given in (4.2), then $F = T \circ \varphi^{-1} \circ L^{-1} \circ S^{-1}$ is a member of $\mathfrak{L}(\mathbb{C}^n)$ taking ∞ into ∞. Therefore U is the full rank affine image, under F, of Ω. ■

The image of the unit circle under a linear fractional transformation of one complex variable is always a circle on the Riemann sphere $\mathbb{C}_\infty$. It does not appear that there is an appropriate metric on $\mathbb{CP}^n$ for which the image of the unit sphere in $\mathbb{C}^n$ is always a "sphere" in $\mathbb{CP}^n$. The natural generalization of the spherical metric to $\mathbb{CP}^n$ is the Fubini-Study metric (see [7]). If one calculates the image of the unit sphere of $\mathbb{C}^n$ under the transformations given by (1.2) and (4.2), the distance from the "center" of the image in $\mathbb{CP}^n$ under this metric varies with $|z_1|$.

5. SYMMETRIC POINTS

With linear fractional transformations of the plane, symmetric points provide a simple method to determine a transformation between domains in $\mathcal{A}$. We desire an analog in higher dimensions. The following lemma and corollary provide a useful start.

LEMMA 5.1 *Let $T \in \mathfrak{L}(\mathbb{C}^n)$, and let $L \subseteq \mathbb{C}^n$ be a complex line such that $T(L) \nsubseteq \infty$. Then $T(L)$ is a complex line and T maps circles and lines in L onto circles and lines in $T(L)$.*

Proof The preservation of affine sets under linear fractional transformations verifies that $T(L)$ is a complex line. Without loss of generality, suppose that $L = \{\lambda e_1: \lambda \in \mathbb{C}\}$, and that $w_0 = T(0)$ and $w_1 = T(e_1)$ are finite. Define $\varphi : \mathbb{C} \to \mathbb{C}_\infty$ implicitly by the equation

$$T(\lambda e_1) = (1 - \varphi(\lambda))w_0 + \varphi(\lambda)w_1. \tag{5.1}$$

The result will follow due to a one-variable argument provided that $\varphi \in \mathfrak{L}(\mathbb{C})$.

If we assume T has the form (2.1), then (5.1) becomes

$$\frac{\lambda A_1 + a}{\alpha + \lambda \overline{b}_1} = (1 - \varphi(\lambda))\frac{a}{\alpha} + \varphi(\lambda)\frac{A_1 + a}{\alpha + \overline{b}_1},$$

where A_1 is the first column of A. Direct calculation then shows that

$$\varphi(\lambda) = \frac{(\alpha + \overline{b}_1)\lambda}{\alpha + \overline{b}_1\lambda},$$

as desired. ■

Corollary 5.2 *If $U \in \mathcal{A}$ and L is a complex line in $\mathbb{C}^n$ intersecting U, then $L \cap \partial U$ is a circle or a line.*

We combine this with our notion of symmetric points in the plane to develop a definition for $\mathbb{C}^n$ in a natural way.

Definition 5.3 Let $U \in \mathcal{A}$, and fix $z_0 \in U$. A point $z \in \mathbb{C}^n$ is *symmetric* to z_0 with respect to ∂U provided that z is symmetric to z_0, in the one-variable sense, with respect to the circle or line $L \cap \partial U$, where L is the complex line through z and z_0. Denote the set of all $z \in \mathbb{C}^n$ symmetric to z_0 with respect to ∂U by $\Sigma(z_0; \partial U)$.

The set of symmetric points has the following structure.

Theorem 5.4 *Let $U \in \mathcal{A}$ and $z_0 \in U$. The set $\Sigma(z_0; \partial U)$ is either empty or an $(n-1)$-dimensional affine set in $\mathbb{C}^n$.*

It should be noted that symmetric points in ∞ are not being considered. Clearly, if $\Sigma(z_0; \partial U) = \emptyset$, then there are symmetric points in ∞. Of course, even when $\Sigma(z_0; \partial U) \neq \emptyset$, the set $\overline{\Sigma(z_0; \partial U)} \cap \infty$ in $\mathbb{CP}^n$ contains symmetric points (provided that $n \geq 2$). However, the abundance of symmetric points makes considering infinite symmetric points unnecessary in this case.

Proof of Theorem 5.4 Since linear fractional transformations send affine sets to affine sets and preserve symmetric points, it suffices to assume $U = B$. Obviously, $\Sigma(0; \partial B) = \emptyset$ by definition, and thus suppose $z_0 \neq 0$. By rotation by a unitary operator, assume $z_0 = re_1$ for some $r \in (0, 1)$.

For each $u \in \partial B$, define $L_u = \{z_0 + \lambda u\colon \lambda \in \mathbb{C}\}$. To determine the point symmetric to z_0 in $L_u \cap \partial B$, we must find the circle in $\mathbb{C}$ given by $C = \{\lambda \in \mathbb{C}\colon \|z_0 + \lambda u\| = 1\}$. First note that

$$\|z_0 + \lambda u\|^2 = r^2 + 2r\,\mathrm{Re}\,\lambda u_1 + |\lambda|^2 = |\lambda + r\overline{u}_1|^2 + r^2(1 - |u_1|^2).$$

It follows that

$$C = \left\{ -r\overline{u}_1 + e^{i\theta}\sqrt{1 + r^2(|u_1|^2 - 1)}\colon\ 0 \le \theta \le 2\pi \right\}.$$

If $\lambda_0 \in \mathbb{C}$ is symmetric to 0 with respect to C, then λ_0 lies in the direction of $\overline{u}_1$ from the center of C and obeys the distance equation

$$|\lambda + r\overline{u}_1| = \frac{1 + r^2(|u_1|^2 - 1)}{r|u_1|}.$$

Therefore

$$\lambda_0 = -r\overline{u}_1 + \frac{\overline{u}_1}{|u_1|} \cdot \frac{1 + r^2(|u_1|^2 - 1)}{r|u_1|} = \frac{1 - r^2}{ru_1}.$$

It follows that $z_0 + \lambda_0 u$ is symmetric to z_0 with respect to the circle $L_u \cap \partial B$.

We have found that

$$\Sigma(z_0; \partial B) = \left\{ z_0 + \left(\frac{1 - r^2}{ru_1} \right) u\colon\ u \in \partial B \right\},$$

where $u_1 = 0$ will give symmetric points in ∞. Replacing $u \in \partial B$ with γu, where $|\gamma| = 1$, yields the same symmetric point, and therefore assume that $u_1 > 0$. Now $\{u \in \partial B\colon u_1 \ge 0\}$ is homeomorphic to the closed unit ball $\overline{B}_{n-1}$ of $\mathbb{C}^{n-1}$ due to the mapping

$$\overline{B}_{n-1} \ni \zeta \mapsto \left(\sqrt{1 - \|\zeta\|^2}, \zeta \right) \in \partial B.$$

Therefore

$$\Sigma(z_0; \partial B) = \left\{ z_0 + \frac{1 - r^2}{r} \left(1, \frac{\zeta}{\sqrt{1 - \|\zeta\|^2}} \right)\colon\ \zeta \in B_{n-1} \right\},$$

which is an $(n-1)$-dimensional affine set. ■

With the help of Theorem 3.2, we use Theorem 5.4 to find a transformation in $\mathfrak{L}(\mathbb{C}^n)$ mapping U onto V for given $U, V \in \mathcal{A}$. Select $z_1 \in U$ such that $\Sigma(z_1; \partial U) \neq \emptyset$. The affine set $\Sigma(z_1; \partial U)$ is then determined by vectors $z_2, \ldots, z_{n+1}$. Choose $z_{n+2} \in \partial U$ that does not lie on the complex lines through z_1 and z_k, $k = 2, \ldots, n+1$. If $w_1, \ldots, w_{n+2} \in \mathbb{C}^n$ are chosen in a similar way with respect to V, then the transformation $T \in \mathfrak{L}(\mathbb{C}^n)$ given in Theorem 3.2 will map U onto V. Moreover, T is unique with respect to the image of the points $z_1, \ldots, z_{n+2}$.

6. APPLICATIONS

We now show how the methods of previous sections can be applied to some mapping problems. Some of these results are developed in [1,2,4–6,9] from a different point of view. The reader should notice that the application of linear fractional transformations gives proofs of these results that are reminiscent of the proofs in the one-variable case, some of which are found in [8].

We begin with an analysis of automorphisms of the Siegel right half-space H given in (1.3). We will find it convenient to write $z = (z_1, \hat{z})$, where $\hat{z} \in \mathbb{C}^{n-1}$, for a vector $z \in \mathbb{C}^n$. Given an operator A, A_k will denote the vector in the kth column of A (as a matrix). The following theorem gives the transformations in $\operatorname{Aut} H$ that fix 0.

THEOREM 6.1 *Let $T \in \mathfrak{L}(\mathbb{C}^n)$. Then $T \in \operatorname{Aut} H$ and $T(0) = 0$ if and only if there exist $c > 0$, $b \in \mathbb{C}^n$ with $\operatorname{Re} b_1 = \|\hat{b}\|^2/4$, and $U \in \mathcal{U}(n-1)$ such that T has the form* (2.1) *with $a = 0$, $\alpha = 1$, and*

$$A = \begin{bmatrix} c^2 & 0^T \\ cU\hat{b}/2 & cU \end{bmatrix}. \tag{6.1}$$

Proof Assuming that $T(0) = 0$ and $T(H) = H$, we know that T may be written in the form (2.1) with $a = 0$ and $\alpha = 1$ and that

$$\operatorname{Re}\left((1 + \langle b, z\rangle) \sum_{k=1}^{n} A_{1k} z_k\right) - \left\| \sum_{k=1}^{n} z_k \hat{A}_k \right\|^2 \geq 0 \tag{6.2}$$

whenever

$$\operatorname{Re} z_1 - \|\hat{z}\|^2 \geq 0, \tag{6.3}$$

with equality in (6.2) and (6.3) occurring simultaneously. Vectors $z \in \mathbb{C}^n$ causing equality in (6.3) depend upon $\hat{z}$ and t in the sense that

$$z = (\|\hat{z}\|^2 + it, \hat{z}), \quad \hat{z} \in \mathbb{C}^{n-1}, \ t \in \mathbb{R}. \tag{6.4}$$

Equality in (6.2) then holds for z as in (6.4). The only terms in (6.2) that are linear in $\hat{z}$ are $\sum_{k=2}^{n} A_{1k} z_k$, and therefore $A_{1k} = 0$ for $k = 2, \ldots, n$. The only term in (6.2) that is linear in t and contains no other variables is $\operatorname{Re} A_{11} it$, and therefore $A_{11} \in \mathbb{R} \backslash \{0\}$. Set $c^2 = A_{11}$, and write

$$A = \begin{bmatrix} c^2 & 0^T \\ \hat{A}_1 & \hat{A} \end{bmatrix}.$$

With z_1 as in (6.4), (6.2) becomes

$$c^2(\|\hat{z}\|^2 + |z_1|^2 \operatorname{Re} b_1 + \operatorname{Re}(z_1\langle \hat{b}, \hat{z}\rangle)) = |z_1|^2 \|\hat{A}_1\|^2 + 2\operatorname{Re}(z_1\langle \hat{A}_1, \hat{A}\hat{z}\rangle) + \|\hat{A}\hat{z}\|^2. \tag{6.5}$$

Equate powers of $\|\hat{z}\|$ in (6.5) to see that $c^2 = \|\hat{A}\hat{z}\|^2/\|\hat{z}\|^2$, and therefore we may assume $c > 0$ and $\hat{A} = cU$ for some $U \in \mathcal{U}(n-1)$. Similarly, we find that $\operatorname{Re} b_1 = \|\hat{A}_1\|^2/c^2$ and $\hat{b} = 2U^*\hat{A}_1/c$. This gives (6.1).

The converse is easily verified. ∎

The closure of H in $\mathbb{CP}^n$ contains exactly one point in ∞, namely $[1 : 0 : \cdots : 0]$. Denote this point by ∞_H. Our characterization of the automorphisms of H that fix 0 can now be used to find those that fix ∞_H.

THEOREM 6.2 *Let $T \in \mathfrak{L}(\mathbb{C}^n)$. Then $T \in \operatorname{Aut} H$ such that $T(\infty_H) = \infty_H$ if and only if there exist $d > 0$, $a \in \partial H$, and $U \in \mathcal{U}(n-1)$ such that $T(z) = Az + a$, where*

$$A = \begin{bmatrix} d^2 & 2d(U^*\hat{a})^* \\ 0 & dU \end{bmatrix}. \tag{6.6}$$

Proof Let S be the generalized inversion (4.2) and let $R \in \operatorname{Aut} H$ be of the type given in Theorem 6.1. It is simple to verify that $S \in \operatorname{Aut} H$ and interchanges 0 and ∞_H. We obtain transformations described in the theorem by taking T to correspond to the operators $\tilde{S}^{-1}\tilde{R}\tilde{S} = \tilde{S}\tilde{R}\tilde{S}$ with various choices of b, c, and U in the definition of R. We then write T in the desired form by replacing c with $1/d$, b_1 with $\overline{a}_1/d^2$, and $\hat{b}$ with $2U^*\hat{a}/d$. ■

Theorems 6.1 and 6.2 combine in the following natural way.

THEOREM 6.3 *Let $T \in \mathfrak{L}(\mathbb{C}^n)$. Then $T \in \operatorname{Aut} H$ such that $T(0) = 0$ and $T(\infty_H) = \infty_H$ if and only if there exists $c > 0$ and $U \in \mathcal{U}(n-1)$ such that $T(z) = Az$, where*

$$A = \begin{bmatrix} c^2 & 0^T \\ 0 & cU \end{bmatrix}. \tag{6.7}$$

The points 0 and ∞_H are the only fixed points of T if and only if $c \neq 1$.

Proof Only the last statement of the theorem needs verification. If $c = 1$, then λe_1 is a fixed point of T in $\overline{H}$ for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \geq 0$. Conversely, if $z \in \operatorname{Fix} T$, then (6.7) gives that $c^2 z_1 = z_1$. This is possible for $z_1 \notin \{0, \infty\}$ only if $c = 1$. ■

The following theorem involving the iterates $\{T^k\}$ of $T \in \mathfrak{L}(\mathbb{C}^n)$ is immediate. (We define $T^1 = T$ and $T^k = T \circ T^{k-1}$ for $k = 2, 3, \ldots$.)

THEOREM 6.4 *If $T \in \operatorname{Aut} H$ fixes 0 and ∞_H and has no other fixed points in $\overline{H}$, then the iterates of T converge uniformly on compact sets to one of the two fixed points.*

Proof Due to Theorem 6.3, we have

$$T^k(z) = (c^{2k} z_1, c^k U^k \hat{z})$$

for all $k = 1, 2, \ldots$, where $c > 0$, $c \neq 1$. If $c \in (0, 1)$, then clearly $T^k \to 0$ uniformly on compact sets. If $c > 1$, then

$$T^k(z) = [c^{2k} : c^k U^k \hat{z} : 1] = \left[1 : \frac{U^k \hat{z}}{c^k} : \frac{1}{c^{2k}}\right] \to \infty_H,$$

as wished. ■

We will use Theorem 6.4 to consider fixed points of members of Aut B. Recall the notation in (1.1). For these important transformations, we have fixed point information beyond that given in Theorem 3.3.

THEOREM 6.5 *If $T \in \mathrm{Aut}\, B$ with $T(ru) = 0$, ($u \in \partial B$, $r \in [0, 1)$), and if $z \in \overline{B} \cap \mathrm{Fix}\, T$, then either z is an isolated fixed point in ∂B or z is on the cylinder $\{z \in \overline{B}$: $P_u z = (1 + \gamma s_r)u/r,\ |\gamma| = 1\}$.*

Proof By the identity in Theorem 2.2.5 of [6], we have

$$1 - \|T(z)\|^2 = \frac{s_r^2(1 - \|z\|^2)}{|1 - r\langle z, u\rangle|^2}, \quad z \in \overline{B}.$$

Therefore, if $z \in \mathrm{Fix}\, T$ and $\|z\| \neq 1$, then $|1 - r\langle z, u\rangle| = s_r$. The result readily follows. ■

The following two theorems are from [4,9], respectively.

THEOREM 6.6 *If $T \in \mathrm{Aut}\, B$ and T fixes three distinct points of ∂B, then T fixes a point of B.*

THEOREM 6.7 *If $T \in \mathrm{Aut}\, B$ and T has two fixed points on ∂B and no fixed points on B, then the iterates of T converge uniformly on compact sets to one of the fixed points on ∂B.*

The argument for Theorem 6.7 is now almost identical to the proof for the $n = 1$ case given in [8]. Originally, techniques involving slicing and one-variable arguments were used.

Proof of Theorems 6.6 and 6.7 Let $S \in \mathfrak{L}(\mathbb{C}^n)$ map B onto H with two points of $\partial B \cap \mathrm{Fix}\, T$ taken to 0 and ∞_H. Then $F = S \circ T \circ S^{-1}$ is in Aut H and has 0 and ∞_H as fixed points in $\overline{H}$. By Theorem 6.3, if T has a third fixed point on ∂B, then F fixes e_1, proving Theorem 6.6. If T has no other fixed points in $\bar{B}$, then Theorem 6.7 follows from Theorem 6.4 since $F^k = S \circ T^k \circ S^{-1}$. ■

By mapping B onto H in a way similar to the proof of Theorem 6.7 and using Theorem 6.3, we see that the subgroup of Aut B of automorphisms that fix two distinct boundary points is isomorphic to $\mathbb{R}_+ \times \mathcal{U}(n-1)$, where $\mathbb{R}_+$ is the multiplicitive group of positive real numbers. Likewise, using Theorem 6.2, we see that the subgroup of Aut B of automorphisms that fix one given boundary point is isomorphic to the semidirect product $\mathcal{H}(n) \times_\theta (\mathbb{R}_+ \times \mathcal{U}(n-1))$, where $\mathcal{H}(n)$ is the Heisenburg group in $\mathbb{C}^n$ (see [6]) and $\theta : \mathbb{R}_+ \times \mathcal{U}(n-1) \to \mathrm{Aut}\, \mathcal{H}(n)$ is a natural homomorphism.

We conclude with two examples of these results concerning fixed points of automorphisms of B. For each, let $T = U \circ \varphi_{re_1}$ for some $r \in [0, 1)$. (We will define U separately in each example.) Set $\zeta = -s_r/(1 - rz_1)$. From Theorem 6.5, a fixed point z of T lines on the cylinder if and only if $|\zeta| = 1$. Take note of the identities

$$z_1 = \frac{s_r + \zeta}{r\zeta}, \quad \frac{r - z_1}{1 - rz_1} = \frac{1 + s_r\zeta}{r}.$$

Example 6.8 Fix an integer $2 \le k \le n$ and let $U \in \mathcal{U}(n)$ be the permutation operator defined on the basis vectors by $Ue_1 = e_k$, $Ue_j = e_{j-1}$ for $2 \le j \le k$, and $Ue_j = e_j$ for $k+1 \le j \le n$ (with obvious adjustment if $k = n$). Setting $T(z) = z$ gives the following equation in ζ:

$$\zeta^{k+1} + \frac{\zeta^k}{s_r} - \frac{\zeta}{s_r} - 1 = 0. \tag{6.8}$$

One solution of (6.8) is $\zeta = 1$, which leads to the fixed points $\{z \in \overline{B}: z_j = (1+s_r)/r : j = 1, \ldots, k\}$. This is an affine set of fixed points of dimension $n-k$ outside of $\overline{B}$. By Theorem 6.5, the remaining solutions of (6.8) must lead to isolated fixed points on ∂B. From Theorem 6.6, there can be at most two solutions of (6.8) that satisfy $|\zeta| \ne 1$. In fact, it is easy to check that all solutions of (6.8) lie on the unit circle when k is odd, and two solutions are not on the circle when k is even.

Example 6.9 Let $U \in \mathcal{U}(n)$ be given by $Uz = (z_1, -\hat{z})$. This results in the equation

$$\zeta^2 - 1 = 0. \tag{6.9}$$

Corresponding to the root $\zeta = -1$ is the $(n-1)$-dimensional affine set given by $z_1 = (1-s_r)/r$ that intersects B. Theorem 3.3 then implies that the fixed point corresponding to the root $\zeta = 1$ must be isolated. It is the point $(1+s_r)e_1/r$.

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References

[1] M. Abate (1989). *Iteration Theory of Holomorphic Maps on Taut Manifolds*. Mediterranean Press, Rende.
[2] C.C. Cowen and B.D. MacCluer (1995). *Composition Operators on Spaces of Analytic Functions*. CRC Press, Boca Raton, FL.
[3] A. Denjoy (1926). Sur l'itération des fonctions analytiques. *C.R. Acad. Sci. Paris*, **182**, 255–257.
[4] T.L. Hayden and T.J. Suffridge (1971). Biholomorphic maps in Hilbert space have a fixed point. *Pacific J. Math.*, **38**, 419–422.
[5] B.D. MacCluer (1983). Iterates of holomorphic self-maps of the open unit ball in $\mathbb{C}^n$. *Mich. Math. J.*, **30**, 97–106.
[6] W. Rudin (1980). *Function Theory in the Unit Ball of* $\mathbb{C}^n$. Springer-Verlag, New York.
[7] B.V. Shabat (1992). *Introduction to Complex Analysis, Part II, Functions of Several Variables*. American Mathematical Society, Providence.
[8] A.L. Shields (1964). On fixed points of commuting analytic functions. *Proc. Amer. Math. Soc.*, **15**, 703–706.
[9] T.J. Suffridge (1974). Common fixed points of commuting holomorphic maps of the hyperball. *Mich. Math. J.*, **21**, 309–314.
[10] J. Wolff (1926). Sur l'itération des fonctions holomorphes dans une région, et dont valeurs appartiennent à cette région. *C.R. Acad. Sci. Paris*, **182**, 42–43.
[11] J. Wolff (1926). Sur l'itération des fonctions bournées. *C.R. Acad. Sci. Paris*, **182**, 200–201.